Sample Path Bounds for Long Memory FBM Traffic

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Abstract—Fractional Brownian motion (fBm) emerged as a useful model for self-similar and long-range dependent Internet traffic. Asymptotic, respectively, approximate performance measures are known from large deviations theory for single queuing systems with fBm traffic. In this paper we prove a rigorous sample path envelope for LRD fBm traffic based on our outcome that overflow probabilities for fBm traffic have a Weibull tail. We find that both approaches agree in their outcome that systems with fBm traffic. In this paper we prove a rigorous (non-asymptotic) sample path bound was proven to be logarithmically asymptotical. The challenge of the fBm traffic model in case of LRD is that its variance $\sigma^2 t^{2H}$ grows superlinearly in $t$, i.e. the traffic is heavily bursty with burst periods that are more likely to be sustained for a long time. These properties make the calculation of performance bounds for fBm traffic hard.

The backlog process $B(t)$ at a lossless work-conserving constant rate server with capacity $C$ is given e.g. in [17] as

$$B(t) = \sup_{\tau \in [0,t]} \{A(\tau, t) - C(t - \tau)\}.$$  

The difficulty behind the analysis of a statistical bound $b$ for the steady state backlog $B$, i.e. letting $t \to \infty$, is to find the value $\tau^\star$ that achieves the supremum in

$$P[B > b] = P\left[\sup_{\tau \in [0,t]} \{A(\tau, t) - C(t - \tau)\} > b\right]$$

since $\tau^\star$ is a random variable, see [14] for explanation.

Large deviations theory is frequently used to analyze the asymptotic decay rate of overflow probabilities [9], [17]. It makes use of the principle of the largest term stating that

$$P[B > b] \approx \sup_{\tau \in [0,t]} P[A(\tau, t) - C(t - \tau) > b]$$

where the term on the right hand side is only a lower bound. For fBm traffic at a server with capacity $C$ the asymptotic bound $P[B > b] = \varepsilon_a$ for $b \to \infty$ has overflow probability [9]

$$\varepsilon_a = \exp\left(-\frac{1}{2\sigma^2} \left(\frac{C - \lambda}{H} \right)^{2H} \left(\frac{b}{1 - H} \right)^{2 - 2H}\right).$$

The overflow probability has a Weibull tail that simplifies to an exponential distribution for the special case $H = \frac{1}{2}$. The backlog bound was proven to be logarithmically asymptotical.

The large deviations result (4) agrees with a solution deduced for the largest term (3) in [7], [8]. The derivation makes use of the Gaussian distribution of the increments and yields the approximation $P[B > b] \approx \varepsilon_a$ where $\varepsilon_a$ is identical to (4).
In [18] it is shown that the largest term (3) obtained at the critical time scale $\tau^*$ dominates all other terms of (2). Asymptotes for $b \to \infty$ of max, sum, and product approximations of (2) are derived providing further evidence for the Weibull tail.

The proof of the large deviations theory builds on the Gärtner-Ellis condition which establishes a direct relation to the effective bandwidth of a traffic flow [9]. The theory of effective bandwidths, e.g. [11], [17], is a major tool for the analysis of traffic flows as it gives a measure for resource requirements at different time scales. The effective bandwidth of a flow $\alpha(\theta, t) = \frac{1}{\theta} \log E \left[ e^{\theta \lambda(t)} \right]$ lies between its average and peak rate depending on the parameter $\theta > 0$. For fBm it holds that

$$\alpha(\theta, t) = \lambda + \frac{\theta \sigma^2}{2} t^{2H-1}.$$  \hspace{1cm} (5)

In case of $H \in (\frac{1}{2}, 1)$ the effective bandwidth of fBm traffic exhibits a continuous growth in $t$ due to LRD [11].

In [14] a connection between effective bandwidths and effective envelopes is established. In contrast to asymptotic results for large buffers from large deviations theory, effective envelopes [12], [13], [14] in conjunction with the stochastic network calculus [19], [20] can provide non-asymptotic performance bounds. Effective envelopes $E(t - \tau)$ are statistical upper bounds on the arrivals $A(\tau, t)$ of the form

$$P[A(\tau, t) - E(t - \tau) > 0] \leq \varepsilon_p.$$  \hspace{1cm} (6)

It constitutes the natural shape of an fBm envelope in the sense that it is the optimal solution that can be derived from its moment generating function using Chernoff’s bound [14].

The definition of effective envelope is point-wise in the sense that it can be violated at each point in time with overflow probability $\varepsilon_p$. Applying the approximation by the largest term (3) the envelope (6) is used in [15] to recover the backlog bound from large deviations theory (4).

In contrast, the derivation of performance bounds using the stochastic network calculus builds on sample path arguments, such as the backlog bound (2), and requires a bound for $A(\tau, t)$ for all $\tau \in [0, t]$, i.e. a sample path envelope of the form

$$P \left[ \sup_{\tau \in [0, t]} \{A(\tau, t) - E(t - \tau)\} > 0 \right] \leq \varepsilon_s.$$  \hspace{1cm} (7)

Under the assumption of a time scale limit $T$ such sample path envelopes are constructed in [14] using Boole’s inequality, i.e. by summing the constant point-wise overflow probabilities $\varepsilon_p$ over all $t \in [0, T]$ resulting in $\varepsilon_s = T \varepsilon_p$. The time scale in this context can be regarded as a constraint on the duration of busy periods. In case of fBm the duration of busy periods has, however, been found to grow extremely fast with $H$ [21].

Methods for construction of sample path envelopes that do not require a priori assumptions on the relevant time scale have been developed in [22], [13], [19]. The general approach is to use a point-wise envelope with parameter $b$

$$P \left[ A(\tau, t) - E(t - \tau) > b \right] \leq \varepsilon_p(b)$$  \hspace{1cm} (8)

that has a decaying and integrable overflow profile $\varepsilon_p(b)$, i.e. $\int_0^\infty \varepsilon_p(b)db$ is finite. Typically, when constructing a sample path envelope $b$ is substituted by a slack rate $\mu \cdot (t - \tau)$. The slack rate relaxes the envelope such that $\varepsilon_p$ decreases with increasing interval width $(t - \tau)$. Finally, taking Boole’s inequality over all $\tau$ to derive the sample path overflow probability $\varepsilon_s$ (7) translates to integrating $\int_0^\infty \varepsilon_p(b(t - \tau))d\tau$ that remains finite for all $t \geq 0$ including $t \to \infty$.

The construction of sample path envelopes for fBm traffic is, however, not straightforward since the overflow profile of fBm is not easily integrable. A simplistic approach is proposed in [13] where it is argued that the Weibull tail (4) implies an envelope for fBm traffic. Such an envelope is, however, based on the approximation by the largest term (3). fBm sample path envelopes are provided in [16] and more recently in [23].

### III. FBM Sample Path Envelope

In this section we derive a sample path envelope for fBm traffic. We use a discrete time model, i.e. time $t$ is a dimensionless counter of slots each of fixed duration. Accordingly, for the fBm traffic model (1) the rate $\lambda$ is given in bits per time slot and the increment $\Delta(t)$ in bits. Subscripted $\varepsilon_p$, $\varepsilon_s$, and $\varepsilon_a$ denote point-wise, sample path, and approximate or asymptotic overflow probabilities, respectively, see Sect. II.

The difficulty of deriving a sample path envelope for fBm traffic is due to the intended integrability of the point-wise overflow probability $\int_0^\infty \varepsilon_p(t)dt$ as introduced in Sect. II. To this end, we consider the point-wise envelope (6) with a time-dependent overflow probability $\varepsilon_p(t)$.

Here, we have to choose the point-wise overflow probability $\varepsilon_p(t)$ in such a way that it is integrable. At the same time $E(t) = \lambda t + \sqrt{2 \log \varepsilon_p} \sqrt{t^H}$. It constitutes the natural shape of an fBm envelope in the sense that it is the optimal solution that can be derived from its moment generating function using Chernoff’s bound [14].

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$$P \left[ \sup_{\tau \in [0, t]} \{A(\tau, t) - E(t - \tau)\} > 0 \right] \leq \varepsilon_s.$$  \hspace{1cm} (7)

Under the assumption of a time scale limit $T$ such sample path envelopes are constructed in [14] using Boole’s inequality, i.e. by summing the constant point-wise overflow probabilities $\varepsilon_p$ over all $t \in [0, T]$ resulting in $\varepsilon_s = T \varepsilon_p$. The time scale in this context can be regarded as a constraint on the duration of busy periods. In case of fBm the duration of busy periods has, however, been found to grow extremely fast with $H$ [21].

Methods for construction of sample path envelopes that do not require a priori assumptions on the relevant time scale have been developed in [22], [13], [19]. The general approach is to use a point-wise envelope with parameter $b$

$$P \left[ A(\tau, t) - E(t - \tau) > b \right] \leq \varepsilon_p(b)$$  \hspace{1cm} (8)
The parameter $\beta$ relaxes the envelope. The case $\beta = 0$ coincides with (6).

Theorem 1 (FBM Sample Path Envelope). Given fBm traffic with mean rate $\lambda$, standard deviation $\sigma$, and LRD Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$,

$$E(t) = \lambda t + \sqrt{-2 \log \sigma} t^{H+\beta}$$

satisfies the definition of sample path envelope (7) with overflow probability

$$\varepsilon_s = \frac{\Gamma\left(\frac{1}{2\beta}\right)}{2\beta(-\log \eta)^{\frac{1}{2\beta}}}$$

where $\beta \in (0, 1-H)$ and $\eta \in (0, 1)$ are free parameters.

Proof: Assuming stationarity of the arrivals and letting $t \rightarrow \infty$ the sample path envelope (7) can be written as

$$P\left[\sup_{\tau \geq 0} \{A(\tau) - E(\tau)\} > 0\right] \leq \varepsilon_s. \tag{9}$$

The overflow probability is defined by a union of events that can be estimated by application of Boole’s inequality

$$P\left[\sup_{\tau \geq 0} \{A(\tau) - E(\tau)\} > 0\right] \leq \sum_{\tau = 1}^{\infty} P[\{A(\tau) > E(\tau)\}$$

where we used the fact that the overflow probability at $\tau = 0$ is trivially zero since by definition $A(0) = E(0) = 0$.

Applying Chernoff’s bound and using the effective bandwidth of fBm (5) we have for any $\theta > 0$

$$P[\{A(\tau) > E(\tau)\}] \leq e^{-\theta E(\tau)} E[e^{\theta A(\tau)}] = e^{-\theta E(\tau)} e^{\theta \lambda \tau + \frac{2 \eta^2 \sigma^2}{\beta} \tau^{2H}}.$$

Inserting $E(\tau) = \lambda \tau + \sqrt{-2 \log \sigma} \tau^{H+\beta}$ and minimizing over $\theta > 0$ yields $\theta = \frac{1}{\sigma} \sqrt{-2 \log \eta} \tau^{H+\beta}$ and by insertion

$$P[\{A(\tau) > E(\tau)\}] \leq \eta^{2\beta}.$$

Summing the point-wise overflow probabilities gives

$$\sum_{\tau = 1}^{\infty} \eta^{2\beta} \leq \int_0^\infty \eta^{2\beta} d\tau = \frac{\Gamma\left(\frac{1}{2\beta}\right)}{2\beta(-\log \eta)^{\frac{1}{2\beta}}}$$

where we used that $\eta^{2\beta}$ is monotonically decreasing in $\tau$ to estimate each summand indexed by $\tau$ by an integral over $(\tau - 1, \tau]$. Finally, we applied Lem. 1 from [16].

Fig. 2(a) compares the point-wise overflow probability $\varepsilon_p(t) = \eta^{2\beta}$ of the envelope from Fig. 1(a) for $\beta = 0.04$ with simulation results obtained from $10^9$ fBm sample paths generated with Matlab. The overflow probabilities are computed from Chernoff’s bound and hence are conservative. Fig. 2(b) shows the overflow probability for sample paths of length $t$. The simulation results are obtained by counting the number of sample paths that violate the envelope at least once in $[0, t]$. To derive the analytical bound we sum the point-wise overflow probabilities over all $\tau \in [0, t]$. The dashed horizontal line shows the overflow probability of the sample path envelope from Th. 1 that holds for sample paths of any length $t \geq 0$, i.e. the sum of the point-wise overflow probabilities stays finite for all $t$ and converges to the dashed line for $t \rightarrow \infty$. 

![FBM envelopes according to (6) with $\varepsilon_p(t) = \eta^{2\beta}$.](image1)

![Point-wise overflow probability $\varepsilon_p(t) = \eta^{2\beta}$ vs. simulation results. The overflow probabilities are conservative due to the use of Chernoff’s bound.](image2)

![Sample path overflow probability $\varepsilon_s$ vs. simulation results for sample paths of length $t$. The overflow probabilities are strict upper bounds due to the sample path argument using Boole’s inequality. For $t \rightarrow \infty$ the sample path overflow probability stays finite and converges to the result obtained from Th. 1.](image3)
IV. PERFORMANCE BOUNDS

We use the sample path envelope from Th. 1 to derive performance bounds for a server fed with fBm traffic.

**Theorem 2 (Backlog and Delay Bound).** Consider a lossless work-conserving constant rate server with capacity $C$ fed with fBm traffic as in Th. 1 where $C > \lambda$. The steady state backlog $B$ is bounded by $b$ subject to the overflow probability $\varepsilon_s$

$$P[B > b] \leq \varepsilon_s = \frac{\Gamma(\frac{1}{2\beta})}{2\beta(-\log \eta)_{\frac{1}{2\beta}}}$$

where $\beta \in (0, 1 - H)$ is a free parameter and

$$\eta = \exp\left(-\frac{1}{2\sigma^2} \frac{(C - \lambda)^2(H + \beta)}{(H + \beta)} \left(\frac{b}{1 - (H + \beta)}\right)^{2 - 2(H + \beta)}\right).$$

The steady state delay under first-come first-serve (fcfs) scheduling $W$ is bounded by $P[W > b/C] \leq \varepsilon_s$.

**Proof:** Assuming stationarity and letting $t \to \infty$ we obtain the steady state backlog from (2) as

$$P[B > b] = P\left[\sup_{\tau \geq 0} \{A(\tau) - C\tau\} > b\right].$$

The expression is a special case of sample path envelope (9).

Define $E(t)$ as in Th. 1. If $E(t) \leq b + Ct$ for all $t \geq 0$ then

$$P[B > b] \leq P\left[\sup_{\tau \geq 0} \{A(\tau) - E(\tau)\} > 0\right] \leq \frac{\Gamma(\frac{1}{2\beta})}{2\beta(-\log \eta)_{\frac{1}{2\beta}}}.$$

Given $b$ and $C$ we derive the largest envelope that satisfies the constraint $E(t) \leq b + Ct$ for all $t \geq 0$. To this end, we first find $t = \tau^*$ that minimizes the vertical distance between $E(t)$ and $b + Ct$. Hence, $\tau^*$ is the solution of $\partial E(t)/\partial t = C$. Then, we choose the parameter $\eta \in (0, 1)$ such that $E(t)$ and $b + Ct$ are tangent to each other at $\tau^*$, i.e. $E(\tau^*) = b + C\tau^*$. From the first requirement we derive $\tau^*$ as

$$\tau^* = \left(\frac{\sqrt{-2\log \eta}\sigma(H + \beta)}{C - \lambda}\right)^{\frac{1}{1-H+\beta}}. \quad (10)$$

By insertion of $\tau^*$ into the condition $E(\tau) = b + C\tau$ the optimal parameter $\eta$ follows as given in Th. 2.

The delay bound can be derived as the maximum horizontal distance of $E(t)$ and $Ct$ using the same basic steps. ■

We recover previous results for $\beta = 0$. In this case the optimal parameter $\eta$ in Th. 2 equals the overflow probability of the known asymptotic backlog bound (4) that has been derived using the approximation by the largest term (3). Similarly, $\eta$ is the point-wise overflow probability of the fBm envelope $E(t)$ from (6), where $E(t)$ is the largest envelope that is smaller than $b + Ct$ for all $t \geq 0$. Since $E(t)$ is tangent to $b + Ct$ at $\tau^*$ its overflow probability is generally smaller than $\eta$ except at $\tau^*$ where it attains its maximum that equals $\eta$. Hence, $\tau^*$ is the most probable point for violation of $b + Ct$. Inserting $\eta$ from Th. 2 into $\tau^*$ recovers the critical time-scale in [7].

Fig. 3 displays backlog bounds for a server with capacity $C = 1$ Gb/s fed with fBm traffic with parameters $\lambda = 0.5$ Gb/s, $\sigma = 0.25$ Gb/s, and $H = 0.75$. The overflow probabilities are obtained from Th. 2 using sample path arguments (solid line) and from (4) using the principle of the largest term (dashed line). Both results agree in the slower than exponential decay that is due to LRD. The absolute values differ, however, by a factor of about 1.75 at $\varepsilon = 10^{-9}$ due to the fact that Th. 2 provides a rigorous upper bound whereas (4) is an approximation. The relative difference of the two results decreases for smaller $\varepsilon$. Fig. 4 shows how a variation of the fBm parameters affects the backlog bounds from Fig. 3. Clearly, for larger $\lambda$, $\sigma$, or $H$ both backlog bounds increase. For moderate $H$ the rigorous and the approximate bounds agree well, whereas large $H$ have a huge impact.
To obtain the results shown in Fig. 3 and 4 we optimized the free parameter $\beta \in (0, 1 - H)$ numerically. Typically, we find that $\beta$ is small compared to one. We note that the optimal choice of $\beta$ has significant impact on the decay of $\varepsilon_s$ with $b$.

**Corollary 1 (Weibullian Decay of Overflow Probabilities).** The sample path bound Th. 2 has the same log-asymptotic decay in $b$ as the largest term approximation (4) that is $\log \varepsilon_s \sim \log \varepsilon_a \sim -b^{2-2H}$ where $c$ is a positive constant.

**Proof:** Cor. 1 follows by minimizing $\varepsilon_s$ over $\beta$. To this end we approximate $\varepsilon_s$ from Th. 1, respectively, Th. 2 by

$$\varepsilon_s = \frac{\sqrt{\pi}}{\sqrt{\beta (2e\beta (- \log \eta))^{2\beta}}} \quad \text{for} \quad \beta \ll 1$$

(11)

using Stirling’s formula $\Gamma(x) \approx \sqrt{2\pi x} (x/e)^x$ for $x \gg 1$.

The approximation (11) is exact in the limit $\beta \to 0$.

We use (11) and minor simplifications for small $\beta$ to derive a near optimal solution $\beta^*$. We emphasize that Th. 2 holds for any $\beta \in (0, 1 - H)$ where we approximate $\beta^*$ that minimizes the bound, i.e. inserting $\beta^*$ into Th. 2 yields a rigorous upper bound that is close to its minimal solution.

Assuming $\beta \ll (1 - H)$ and approximating $1 - H - \beta$ by $1 - H$ we compute the derivative of (11) and solve $\partial \varepsilon_s / \partial \beta = 0$ for $\beta$ and find that the minimum of (11) is approached at $\beta = -W(1/(2\log \varepsilon_a))$ where $\varepsilon_a$ coincides with (4). $W(z)$ denotes Lambert’s W function that is the inverse of $z = xe^z$. It is real-valued for $\varepsilon_a < e^{-e/2}$. Since $\beta$ is assumed to be small a good approximation of the optimal solution is

$$\beta^* = \frac{1}{2(-\log \varepsilon_a)}$$

where we estimate the Lambert W function by a linear segment. Note that $\beta^*$ decreases with $b$. We define the quotient

$$\chi = \left( \frac{H^H (1 - H)^{1-H}}{(H + \beta)^{H+\beta}(1 - (H + \beta))^{1-(H+\beta)}} \right)^2$$

that is in $[\frac{1}{2}, 1]$ for $H \in (\frac{1}{2}, 1)$, $\beta \in (0, 1 - H)$ and approaches 1 for small $\beta$. Inserting $\beta^*$ into (11) we find the closed form

$$\varepsilon_s = \frac{b}{C - \lambda} e^1 + \log \chi \sqrt{2\pi(-\log \varepsilon_a)}$$

(12)

as a near optimal solution of Th. 2. Letting $b \to \infty$ it follows that $\beta^* \to 0$, $\log \chi \to 0$, and (12) from Stirling’s formula becomes exact such that $\lim_{b \to \infty} (\log \varepsilon_s / \log \varepsilon_a) = 1$.

Cor. 1 agrees with an earlier log-asymptote in [18] and a more recent result in [23]. Phrasing $\varepsilon_s$ as a function of $\varepsilon_a$ (12) enables us to strengthen significant conclusions on the dimensioning of networks previously obtained from asymptotes and the largest term approximation (4) by sample path arguments. An important example is the relation of spare capacity $C - \lambda$ and buffer size $b$. While for $H = \frac{1}{2}$ halving $C - \lambda$ requires doubling $b$ to achieve constant $\varepsilon_s$ (4) the tradeoff deteriorates for large $H$ [8]. Under LRD spare capacity becomes more important and buffering much less efficient supporting current arguments for reducing router buffers. We note that (12) matches the numerically optimized results in Fig. 3 and 4 almost perfectly, hence we omit reproducing similar graphs.

V. CONCLUSIONS

The contribution of this paper is a sample path envelope for fBm traffic that complements a known approximation using only the largest term. We compute rigorous performance bounds on backlog and delay for fBm traffic at a queuing system. Our sample path envelope and the approximation agree in the Weibullian decay of overflow probabilities. We recover the previous result at the point in time where the violation of an affine envelope by fBm traffic is most probable. The derived sample path envelope facilitates the analysis of fBm queuing networks using the stochastic network calculus.

REFERENCES


