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Spatial Fading Correlation for Local Scattering: A Condition of Angular Distribution

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***Abstract*—Local scattering in the vicinity of the receiver or the transmitter leads to the formation of a large number of multipath components along different spatial angles. A condition of angular distribution, which is valid for only a uniform linear array, is proposed in this paper to justify whether the spatial fading correlation (SFC) remains simple as a Bessel function. If an angular distribution satisfies the condition, a class of angular distributions is revealed and results in simplifying the analysis of the SFC. To demonstrate its practical use, we apply the condition to several angular distributions that are considered in previous works. It is found that cosine and von Mises distributions follow the condition, whereas uniform, Gaussian, and Laplacian distributions do not satisfy the condition and then one needs to calculate the sinusoidal coefficients for the SFC computation.**

***Index Terms*—Antenna array, local scattering, spatial fading correlation.**

I. Introduction

In wireless communications, local scattering around the transmitter or the receiver leads to the formation of a large number of multipath components. In multiantenna communications systems, the receiver features the correlation among the impulse responses of different pair of antenna elements, namely spatial fading correlation (SFC), as an impact on link quality.

The SFC plays an important role in the performance analysis of a wireless communications system, because most of the performance metrics, e.g., bit error probability [1]–[3] and channel capacity [4]–[6], depend on it. Therefore, several works pay attention to the SFC of antenna array [7]–[15]. In [16,17], the SFC of a circular array is derived for uniform, cosine, and Gaussian angular distributions, respectively, which are chosen as the candidates for fitting several measurement results. However, the direct computation of the SFC requires extensive integrations, which are complicated and possibly infeasible for a complex angular distribution.

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In this paper, a condition of the angular distributions is proposed to justify whether the SFC from a linear antenna array remains simple as a Bessel function [18, Ch. 1]. For a class of the angular distributions, the test of the condition requires only differentiations, which are simpler than the integrations that are required in the direct method. This can help to facilitate the analysis of the fading correlation in the wireless channels. It is discovered that the proposed condition defines a group of the angular distributions (see various families of the distributions in [19, Ch. 7] and [20, Ch. 5]). It means that if an angular distribution satisfies the condition, the SFC remains only a simple form for the calculation. To demonstrate its usage in practice, we apply the condition to several angular distributions that are considered in previous works. It is found that the cosine and the von Mises distributions follow the condition, whereas the uniform, the Gaussian, and the Laplacian distributions do not satisfy the condition and then one needs to calculate the sinusoidal coefficients in the SFC computation.

The gap between this work and the previous works can be seen as follows. The SFC in various environments is studied in [7]–[15]. The effects of the SFC on the system performance are investigated in [1]–[6]. In [21], [22, Sec. 2.2.2], and [23], tedious calculation is avoided by approximating the SFC for a small angular spread. The contributions of the paper can be summarized as follows.

- A condition is proposed to test the angular distribution of local scattering in the vicinity of the transmitter or the receiver, which can happen in the multipath channels that take into account spatial angle observed by a uniform linear array.
- The test requires only the differentiations, instead of the integrations. For a class of the angular distributions, the solution of the SFC is known *a priori* as a simple form of the zeroth-order Bessel function. The proposed condition can be applied to any angular distribution, provided that the uniform linear array is taken into account.

For another class of the angular distributions that do not satisfy the proposed condition, we also provide the derivation of the sinusoidal coefficients for $\phi \in (-\pi, \pi]$. Although we consider in this paper the azimuth plane, $\phi \in (-\pi, \pi]$ [18], the scattering over the half circle $\phi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$ [24] as well as the 3-dimensional scattering can be extended in straightforward treatment based on our derivation idea.

Some mathematical notations are involved as follows. $\mathrm{E}_{\phi}\{\cdot\}$ is the expectation with respect to ϕ whose probability density function (pdf) is $p_\phi(\phi)$. $J_0(\cdot)$ and $J_k(\cdot)$ are the zeroth order and the k-th order Bessel functions of the first kind. $I_0(\cdot)$ and $I_k(\cdot)$ are the zeroth order and the k-th order modified Bessel functions of the first kind. The error function $\mathrm{erf}(z)$ is defined as $\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-u^2}\mathrm{d}u$. $(\cdot)^*$ is the conjugate of a complex argument $\cdot$. $\Re(\cdot)$ and $\Im(\cdot)$ are the real and the imaginary components, respectively. $\lfloor\cdot\rfloor$ denotes the integer part of a variable $\cdot$.

II. Spatial Correlation

For the uniform linear array, the time delay at the n-th antenna element is given by

$$\psi_n=\frac{1}{c}d(n-1)\sin(\phi), \tag{1}$$

where c is the wave propagation speed, which is herein equivalent to the speed of light, d is the distance between adjacent antenna elements, and ϕ is the direction of emitting or incoming ray, which is measured from the perpendicular axis of the array. The received signal is composed of a large number of propagating waves along directions, which are characterized by an angular distribution. The correlation between the received signals from the n-th and the $\acute{n}$-th antenna elements, SFC, can be expressed as [18]

$$\rho_{n,\acute{n}}=\mathrm{E}_\phi\left\{\mathrm{e}^{\frac{1}{c}\mathrm{j}2\pi f_0 d(n-\acute{n})\sin(\phi)}\right\}, \tag{2}$$

where f_0 is the operating frequency. The above correlation model takes into account different levels of the received signal energy caused by fading in such a way that the received signal energy along different angles can be compared to an angular distribution. Using the expansions of trigonometry functions (see [25, Sec. 9.1.42-43] and [26, p. 22]), the SFC can be calculated from (see [22, eq. (2.5)])

$$\begin{aligned}\rho_{n,\acute{n}}&=J_0\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)\\&\quad+2\sum_{k=1}^{\infty}J_{2k}\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)c_k\\&\quad+\mathrm{j}J_{2k-1}\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)s_k,\end{aligned} \tag{3}$$

where c_k and s_k are the real and complex sinusoidal coefficients given by

$$c_k=\int_{-\pi}^{\pi}p_\phi(\phi)\cos(2k\phi)\mathrm{d}\phi, \tag{4a}$$

$$s_k=\int_{-\pi}^{\pi}p_\phi(\phi)\sin((2k-1)\phi)\mathrm{d}\phi. \tag{4b}$$

To evaluate the spatial correlation, the calculation incurs the integrations in (4).

III. Angular Distributions

The angular distribution can be derived from statistical distributions, such as cosine distribution [7,27], uniform distribution [9,28,29], Gaussian distribution [8,21], Laplacian distribution [23,30,31], von Mises distribution [32], and etc. (see also other distributions based on geometry, e.g., [33], and scatterer position, e.g., [34]). The pdf of the azimuth angle $\phi\in(-\pi,\pi]$ for the truncated cosine, uniform, truncated Gaussian, truncated Laplacian, and von Mises distributions can be modeled respectively as

$$p_\phi(\phi)=\begin{cases}\frac{1}{\pi}c_\mathrm{c}\cos^l(\phi-\bar{\phi}), & \phi\in(-\frac{1}{2}\pi,\frac{1}{2}\pi], l\in\{2,4,\ldots\};\\ \frac{1}{2\sqrt{3}\sigma_\phi}, & \begin{array}{l}\phi\in(\bar{\phi}-\sqrt{3}\sigma_\phi,\bar{\phi}+\sqrt{3}\sigma_\phi],\\ \bar{\phi}\in(\sqrt{3}\sigma_\phi-\pi,\pi-\sqrt{3}\sigma_\phi);\end{array}\\ \frac{1}{\sqrt{2\pi}\sigma_\phi}c_\mathrm{G}\mathrm{e}^{-\frac{1}{2\sigma_\phi^2}(\phi-\bar{\phi})^2}, & \phi\in(-\pi,\pi];\\ \frac{1}{\sqrt{2}\sigma_\phi}c_\mathrm{L}\mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}|\phi-\bar{\phi}|}, & \phi\in(-\pi,\pi];\\ \frac{1}{2\pi I_0(\kappa)}c_\mathrm{vM}\mathrm{e}^{\kappa\cos(\phi-\bar{\phi})}, & \phi\in(-\pi,\pi],\quad \kappa\geq 0,\end{cases} \tag{5}$$

where $\bar{\phi}$ is the mean of each angular distribution, or nominal angle, and the constants c_c, c_G, c_L, and c_vM are the normalization factors.

A. Cosine Distribution

Note that the cosine distribution supports the angle from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, since if the domain were given from $-\pi$ to π there will be mirror rays on the third and the fourth quadrants. Using the relation $\cos^l(\phi)=\frac{1}{2^l}\left(\binom{l}{\frac{1}{2}l}+2\sum_{k=0}^{\frac{1}{2}l-1}\binom{l}{k}\cos((l-2k)\phi)\right)$ (see [25, p. 31]), the constant c_c can be shown as $c_\mathrm{c}=\frac{1}{\binom{l}{\frac{1}{2}l}}2^l$. From (4) and (5), it can be shown that

$$\begin{aligned}c_k&=\frac{1}{\pi 2^l}c_\mathrm{c}\left(\binom{l}{\frac{1}{2}l}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos(2k\phi)\mathrm{d}\phi\right.\\&\qquad\left.+2\sum_{k=0}^{\frac{1}{2}l-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos(2k\phi)\cos\left((l-2k)(\phi-\bar{\phi})\right)\mathrm{d}\phi\right)\\&=0,\end{aligned} \tag{6}$$

$$\begin{aligned}s_k&=\frac{1}{\pi 2^l}c_\mathrm{c}\left(\binom{l}{\frac{1}{2}l}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin((2k-1)\phi)\mathrm{d}\phi\right.\\&\qquad\left.+2\sum_{k=0}^{\frac{1}{2}l-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin((2k-1)\phi)\cos\left((l-2k)(\phi-\bar{\phi})\right)\mathrm{d}\phi\right)\\&=0.\end{aligned} \tag{7}$$

B. Uniform Distribution

From (4) and (5), it can be shown that

$$\begin{aligned}c_k&=\frac{1}{2\sqrt{3}\sigma_\phi}\int_{\bar{\phi}-\sqrt{3}\sigma_\phi}^{\bar{\phi}+\sqrt{3}\sigma_\phi}\cos(2k\phi)\mathrm{d}\phi\\&=\frac{1}{2k\sqrt{3}\sigma_\phi}\cos(2k\bar{\phi})\sin\left(2k\sqrt{3}\sigma_\phi\right),\end{aligned} \tag{8}$$

$$s_k = \frac{1}{2\sqrt{3}\sigma_\phi}\int_{\bar{\phi}-\sqrt{3}\sigma_\phi}^{\bar{\phi}+\sqrt{3}\sigma_\phi}\sin((2k-1)\phi)\mathrm{d}\phi = \frac{1}{(2k-1)\sqrt{3}\sigma_\phi}\sin((2k-1)\bar{\phi})\sin\left((2k-1)\sqrt{3}\sigma_\phi\right). \tag{9}$$

C. Gaussian Distribution

For the Gaussian distribution, we have $c_\mathrm{G} = \frac{1}{\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}\pi\right)}$. Using the result in [35, p. 109], we can see from (4) and (5) that

$$\begin{aligned} c_k = \frac{1}{4}c_\mathrm{G}\mathrm{e}^{-2k^2\sigma_\phi^2}\Bigg(&\mathrm{e}^{\mathrm{j}2k\bar{\phi}}\Bigg(\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi-\bar{\phi})-\mathrm{j}k\sqrt{2}\sigma_\phi\right) \\ &+\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi+\bar{\phi})+\mathrm{j}k\sqrt{2}\sigma_\phi\right)\Bigg) \\ &+\mathrm{e}^{-\mathrm{j}2k\bar{\phi}}\Bigg(\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi-\bar{\phi})+\mathrm{j}k\sqrt{2}\sigma_\phi\right) \\ &+\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi+\bar{\phi})-\mathrm{j}k\sqrt{2}\sigma_\phi\right)\Bigg)\Bigg). \end{aligned} \tag{10}$$

Using the conjugate property of the error function $\mathrm{erf}(z^*) = \mathrm{erf}^*(z)$, we have

$$\begin{aligned} c_k = \frac{1}{2}c_\mathrm{G}\mathrm{e}^{-2k^2\sigma_\phi^2}\Re\Bigg(&\mathrm{e}^{\mathrm{j}2k\bar{\phi}}\Bigg(\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi+\bar{\phi})+\mathrm{j}k\sqrt{2}\sigma_\phi\right) \\ &+\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi-\bar{\phi})-\mathrm{j}k\sqrt{2}\sigma_\phi\right)\Bigg)\Bigg). \end{aligned} \tag{11}$$

Using the result in [35, p. 109], we can show from (4) and (5) that

$$\begin{aligned} s_k &= \frac{1}{\sqrt{2\pi}\sigma_\phi}c_\mathrm{G}\int_{-\pi}^{\pi}\sin((2k-1)\phi)\mathrm{e}^{-\frac{1}{2\sigma_\phi^2}(\phi-\bar{\phi})^2}\mathrm{d}\phi \\ &= \frac{1}{4}c_\mathrm{G}\mathrm{e}^{-\frac{1}{2}(2k-1)^2\sigma_\phi^2}(-\mathrm{j}) \\ &\quad\Bigg(\mathrm{e}^{\mathrm{j}(2k-1)\bar{\phi}}\Bigg(\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi-\bar{\phi})-\frac{1}{\sqrt{2}}\mathrm{j}(2k-1)\sigma_\phi\right) \\ &\quad+\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi+\bar{\phi})+\frac{1}{\sqrt{2}}\mathrm{j}(2k-1)\sigma_\phi\right)\Bigg) \\ &\quad-\mathrm{e}^{-\mathrm{j}(2k-1)\bar{\phi}}\Bigg(\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi-\bar{\phi})+\frac{1}{\sqrt{2}}\mathrm{j}(2k-1)\sigma_\phi\right) \\ &\quad+\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi+\bar{\phi})-\frac{1}{\sqrt{2}}\mathrm{j}(2k-1)\sigma_\phi\right)\Bigg)\Bigg). \end{aligned} \tag{12}$$

Using the conjugate property of the error function $\mathrm{erf}(z^*) = \mathrm{erf}^*(z)$, we have

$$\begin{aligned} s_k = \frac{1}{2}c_\mathrm{G}\mathrm{e}^{-\frac{1}{2}(2k-1)^2\sigma_\phi^2}\Im\Bigg(&\mathrm{e}^{\mathrm{j}(2k-1)\bar{\phi}} \\ &\Bigg(\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi+\bar{\phi})+\frac{1}{\sqrt{2}}\mathrm{j}(2k-1)\sigma_\phi\right) \\ &+\mathrm{erf}\left(\frac{1}{\sqrt{2}\sigma_\phi}(\pi-\bar{\phi})-\frac{1}{\sqrt{2}}\mathrm{j}(2k-1)\sigma_\phi\right)\Bigg)\Bigg). \end{aligned} \tag{13}$$

D. Laplacian Distribution

For the Laplacian distribution, we have $c_\mathrm{L} = \frac{1}{1-\mathrm{e}^{-\frac{1}{\sqrt{2}\sigma_\phi}\pi}\cosh\left(\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}\right)}$. Using the result in [35, p. 133], we have for $\bar{\phi}\in(-\pi,\pi)$

$$\begin{aligned} c_k &= \begin{cases} \frac{1}{\sqrt{2}\sigma_\phi}c_\mathrm{L}\Bigg(\frac{1}{\frac{1}{\sigma_\phi^2}2+4k^2}\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}}\mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\phi} \\ \quad\left(-\frac{1}{\sigma_\phi}\sqrt{2}\cos(2k\phi)+2k\sin(2k\phi)\right)\Big|_{\phi=\bar{\phi}}^{\pi} \\ \quad+\frac{1}{\frac{1}{\sigma_\phi^2}2+4k^2}\mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}}\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\phi} \\ \quad\left(\frac{1}{\sigma_\phi}\sqrt{2}\cos(2k\phi)+2k\sin(2k\phi)\right)\Big|_{\phi=-\pi}^{\bar{\phi}}\Bigg), & \bar{\phi}\in(-\pi,0); \\ \frac{1}{\sqrt{2}\sigma_\phi}c_\mathrm{L}\Bigg(\frac{1}{\frac{1}{\sigma_\phi^2}2+4k^2}\mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}}\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\phi} \\ \quad\left(\frac{1}{\sigma_\phi}\sqrt{2}\cos(2k\phi)+2k\sin(2k\phi)\right)\Big|_{\phi=-\pi}^{\bar{\phi}} \\ \quad+\frac{1}{\frac{1}{\sigma_\phi^2}2+4k^2}\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}}\mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\phi} \\ \quad\left(-\frac{1}{\sigma_\phi}\sqrt{2}\cos(2k\phi)+2k\sin(2k\phi)\right)\Big|_{\phi=\bar{\phi}}^{\pi}\Bigg), & \bar{\phi}\in[0,\pi), \end{cases} \\ &= \frac{1}{1+2k^2\sigma_\phi^2}c_\mathrm{L}\left(\cos(2k\bar{\phi})-\cosh\left(\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}\right)\mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\pi}\right). \end{aligned} \tag{14}$$

Using the result in [35, p. 133], we have for $\bar{\phi}\in(-\pi,\pi)$

$$s_k = \begin{cases} \frac{1}{\sqrt{2}\sigma_\phi} c_{\mathrm{L}} \left(\frac{1}{\frac{1}{\sigma_\phi^2}2+(2k-1)^2} \mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}} \mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\phi} \left(-\frac{1}{\sigma_\phi}\sqrt{2}\sin((2k-1)\phi) - (2k-1)\cos((2k-1)\phi)\right)\Big|_{\phi=\bar{\phi}}^{\pi} + \frac{1}{\frac{1}{\sigma_\phi^2}2+(2k-1)^2} \mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}} \mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\phi} \left(\frac{1}{\sigma_\phi}\sqrt{2}\sin((2k-1)\phi) - (2k-1)\cos((2k-1)\phi)\right)\Big|_{\phi=-\pi}^{\bar{\phi}} \right), & \bar{\phi}\in(-\pi,0); \\ \frac{1}{\sqrt{2}\sigma_\phi} c_{\mathrm{L}} \left(\frac{1}{\frac{1}{\sigma_\phi^2}2+(2k-1)^2} \mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}} \mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\phi} \left(\frac{1}{\sigma_\phi}\sqrt{2}\sin((2k-1)\phi) - (2k-1)\cos((2k-1)\phi)\right)\Big|_{\phi=-\pi}^{\bar{\phi}} + \frac{1}{\frac{1}{\sigma_\phi^2}2+(2k-1)^2} \mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}} \mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\phi} \left(-\frac{1}{\sigma_\phi}\sqrt{2}\sin((2k-1)\phi) - (2k-1)\cos((2k-1)\phi)\right)\Big|_{\phi=\bar{\phi}}^{\pi} \right), & \bar{\phi}\in[0,\pi), \end{cases}$$
$$= \frac{1}{\sqrt{2}\sigma_\phi\left(\frac{1}{\sigma_\phi^2}2+(2k-1)^2\right)} c_{\mathrm{L}} \left((2k-1)\sinh\left(\frac{1}{\sigma_\phi}\sqrt{2}\bar{\phi}\right) \mathrm{e}^{-\frac{1}{\sigma_\phi}\sqrt{2}\pi} + \frac{1}{\sigma_\phi} 2\sqrt{2}\sin((2k-1)\bar{\phi}) \right). \tag{15}$$

E. Von Mises Distribution

For the von Mises distribution, the expression $\mathrm{e}^{\kappa\cos(\delta_\phi)} = I_0(\kappa) + 2\sum_{k=1}^{\infty} I_k(\kappa)\cos(k\delta_\phi)$ (see [25, eq. (9.6.34)]) results in $c_{\mathrm{vM}} = \frac{1}{\frac{1}{2\pi I_0(\kappa)}\int_{-\pi}^{\pi}\mathrm{e}^{\kappa\cos(\phi-\bar{\phi})}\mathrm{d}\phi} = 1$. Using the relation $\mathrm{e}^{\kappa\cos(\phi)} = I_0(\kappa)+2\sum_{k=1}^{\infty} I_k(\kappa)\cos(k\phi)$ (see [25, eq. (9.6.34)]), we have from (4) and (5)

$$\begin{aligned} c_k &= \frac{1}{2\pi I_0(\kappa)} c_{\mathrm{vM}} \left(I_0(\kappa)\int_{-\pi}^{\pi}\cos(2k\phi)\mathrm{d}\phi + \sum_{k=1}^{\infty} I_k(\kappa) \int_{-\pi}^{\pi}\left(\cos(3k\phi-\bar{\phi})+\cos(k\phi-\bar{\phi})\right)\mathrm{d}\phi \right) \\ &= \frac{1}{\pi I_0(\kappa)} c_{\mathrm{vM}} \sum_{k=1}^{\infty}(-1)^k I_k(\kappa) \left(\sin(2k\pi-\bar{\phi}) - \sin(-2k\pi-\bar{\phi})\right) \\ &= 0, \end{aligned} \tag{16}$$

$$\begin{aligned} s_k &= \frac{1}{2\pi I_0(\kappa)} c_{\mathrm{vM}} \left(I_0(\kappa)\int_{-\pi}^{\pi}\sin((2k-1)\phi)\mathrm{d}\phi + \sum_{k=1}^{\infty} I_k(\kappa) \int_{-\pi}^{\pi}\sin((3k-1)\phi-\bar{\phi}) + \sin((k-1)\phi-\bar{\phi})\mathrm{d}\phi \right) \\ &= -\frac{1}{\pi I_0(\kappa)} c_{\mathrm{vM}} \sum_{k=1}^{\infty}(-1)^k I_k(\kappa) \left(\cos((2k-1)\pi-\bar{\phi}) - \cos(-(2k-1)\pi-\bar{\phi})\right) \\ &= 0. \end{aligned} \tag{17}$$

IV. Angular Distribution Condition

In this section, we establish the condition of the angular distribution from which the SFC reduces to the first term in (3).

Proposition 1 (A condition of angular distribution): Let $p_\phi(\phi)$ be the pdf of an angular distribution, which fully supports the angle $\phi\in(-\pi,\pi]$, i.e., $p_\phi(\phi)\neq 0$ for $\phi\in(-\pi,\pi]$. Let the pdf $p_\phi(\phi)$ be differentiable up to $2M$ times, where $M\in\{1,2,3,\ldots\}$ is an integer. If the condition

$$\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=\pi} = \frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi} \tag{18}$$

holds for $m\in\{0,1,\ldots,2M\}$, the SFC between the n-th and the $\acute{n}$-th antenna elements in (2) yields

$$\rho_{n,\acute{n}} = J_0\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right). \tag{19}$$

Proof: See Appendix A. ■

Some remarks on Proposition 1 are as follows.

- When the condition in (18) is satisfied, the result in (19) is the same as in (2), in which c_k and s_k are zero.
- Proposition 1 holds even when M tends to the infinity, since c_k and s_k can be written as the infinite series of zeros.
- The proposed condition raises the relation of the local scattering to the Clarke/Jakes' model.
- It is obvious that the SFC in (19) under the condition in (18) does not depend on the distribution of associated angles. The result of obtaining only the Bessel function closely coincides with [36], [23, eq. (32)]. The difference from [23,36] is that a particular type of the distribution of ϕ is assumed therein, while in Proposition 1 no certain distribution of the angle is assumed.
- Furthermore, let the angle split into $\phi = \bar{\phi}+\delta_\phi$, where $\bar{\phi}$ is the nominal angle or mean angle, and δ_ϕ is its deviation. The approximation for the small angular spread, $\sin(\bar{\phi}+\delta_\phi)\approx\sin(\bar{\phi})+\delta_\phi\cos(\bar{\phi})$, (see [21], [22, Sec. 2.2.2], [37], and [23]) is not taken into account during the proof.
- From a functional analysis point of view, the condition in (18) may be viewed as a multiple derivative form compared to Lipschitz condition (see [38, p. 32] and [19, p. 320]).

Next we examine whether any of the probability distributions under consideration produces fading that follows the Bessel function from the Jakes' model.

A. Cosine Distribution

For the cosine angles, we have

$$\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi) = \begin{cases} \frac{1}{\pi 2^{l-1}}c_1 \sum_{k=0}^{\frac{1}{2}l-1} \binom{l}{k}\mathrm{j}^{m+1}(l-2k)^m & m\in\{1,3,\ldots\}; \\ \sin\left((l-2k)(\phi-\bar{\phi})\right), & \\ \frac{1}{\pi 2^{l-1}}c_1 \sum_{k=0}^{\frac{1}{2}l-1} \binom{l}{k}\mathrm{j}^{m}(l-2k)^m & m\in\{2,4,\ldots\}. \\ \cos\left((l-2k)(\phi-\bar{\phi})\right), & \end{cases} \tag{20}$$

For $m\in\{1,2,\ldots\}$, the definite values of π and $-\pi$ result in

$$\begin{aligned} &\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &= \begin{cases} \frac{1}{\pi 2^{l-2}}c_1 \sum_{k=0}^{l} \binom{l}{k}\mathrm{j}^{m+1}(l-2k)^m & m\in\{1,3\ldots\}; \\ \cos\left(-(l-2k)\bar{\phi}\right)\sin\left((l-2k)\pi\right), & \\ -\frac{1}{\pi 2^{l-2}}c_1 \sum_{k=0}^{l} \binom{l}{k}\mathrm{j}^{m}(l-2k)^m & m\in\{2,4\ldots\}, \\ \sin\left(-(l-2k)\bar{\phi}\right)\sin\left((l-2k)\pi\right), & \end{cases} \\ &= 0, \end{aligned} \tag{21}$$

which implies that for any $\bar{\phi}$ the condition $\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} = 0$ holds in the case of the cosine distribution.

B. Uniform Distribution

It is clear that the uniform distribution provides $\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi) = 0$. However, the uniform distribution does not follow Proposition 1, because the uniform pdf does not support the full angular existence $\phi\in(-\pi,\pi]$, i.e. there exists ϕ such that $p_\phi(\phi)=0$ for $\phi\in(-\pi,\pi]$.

C. Gaussian Distribution

For the Gaussian distribution, we have

$$\begin{aligned} \frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi) &= \frac{1}{\sqrt{2\pi}\sigma_\phi}c_\mathrm{G}\mathrm{e}^{\frac{1}{2\sigma_\phi^2}(\phi-\bar{\phi})^2} \\ &\quad \sum_{k=0}^{\lfloor\frac{1}{2}m\rfloor} \frac{1}{(\sigma_\phi^2)^{m-k}}c_{m,k}(\phi-\bar{\phi})^{m-2k}, \end{aligned} \tag{22}$$

where $c_{m,k}$ is the coefficient, which also can be drawn from the Hermite polynomial [25, Ch. 22]. It can be shown that

$$\begin{aligned} &\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &= \frac{1}{\sqrt{2\pi}\sigma_\phi}c_\mathrm{G}\sum_{k=0}^{\lfloor\frac{1}{2}m\rfloor}\frac{1}{(\sigma_\phi^2)^{m-k}}c_{m,k}\left(\left(\pi-\bar{\phi}\right)^{m-2k}\right. \\ &\quad \left.\mathrm{e}^{\frac{1}{2\sigma_\phi^2}(\pi-\bar{\phi})^2} - \left(-\pi-\bar{\phi}\right)^{m-2k}\mathrm{e}^{\frac{1}{2\sigma_\phi^2}(-\pi-\bar{\phi})^2}\right). \end{aligned} \tag{23}$$

If $m=1$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} &= \frac{1}{\sqrt{2\pi}\sigma_\phi}\frac{1}{\sigma_\phi^2}c_\mathrm{G}\left(\left(\pi-\bar{\phi}\right)\mathrm{e}^{\frac{1}{2\sigma_\phi^2}(\pi-\bar{\phi})^2}\right. \\ &\quad \left.+\left(\pi+\bar{\phi}\right)\mathrm{e}^{\frac{1}{2\sigma_\phi^2}(\pi+\bar{\phi})^2}\right). \end{aligned} \tag{24}$$

It can be easily seen that if $\bar{\phi}=0$, the derivative $\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi}$ does not remain zero. Therefore, the Gaussian distribution does not satisfy the condition $\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} = 0$.

D. Laplacian Distribution

The derivative of the Laplacian distribution can be written as

$$\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi) = \begin{cases} \frac{1}{\sqrt{2}\sigma_\phi}c_\mathrm{L}\left(\frac{1}{\sigma_\phi}\sqrt{2}\right)^m\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}|\phi-\bar{\phi}|}, & \phi\geq\bar{\phi}; \\ \frac{1}{\sqrt{2}\sigma_\phi}c_\mathrm{L}\left(-\frac{1}{\sigma_\phi}\sqrt{2}\right)^m\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}|\phi-\bar{\phi}|}, & \phi<\bar{\phi}. \end{cases} \tag{25}$$

For $\bar{\phi}\in(-\pi,\pi)$, we have

$$\begin{aligned} \frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} &= \frac{1}{\sqrt{2}\sigma_\phi}c_\mathrm{L}\left(\left(\frac{1}{\sigma_\phi}\sqrt{2}\right)^m\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}(\pi-\bar{\phi})}\right. \\ &\quad \left.-\left(-\frac{1}{\sigma_\phi}\sqrt{2}\right)^m\mathrm{e}^{\frac{1}{\sigma_\phi}\sqrt{2}(\pi+\bar{\phi})}\right). \end{aligned} \tag{26}$$

One can see that the Laplacian distribution, in general, does not provide $\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} = 0$.

E. Von Mises Distribution

Using the relation $\mathrm{e}^{\kappa\cos(\phi)} = I_0(\kappa) + 2\sum_{k=1}^{\infty}I_k(\kappa)\cos(k\phi)$ (see [25, eq. (9.6.34)]), we have

$$\begin{aligned} &\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi) \\ &= \begin{cases} \frac{1}{\pi I_0(\kappa)}\sum_{k=1}^{\infty}(-1)^{\frac{1}{2}(m+1)}k^m & m\in\{1,3,\ldots\}; \\ I_k(\kappa)\sin(k(\phi-\bar{\phi})), & \\ \frac{1}{\pi I_0(\kappa)}\sum_{k=1}^{\infty}(-1)^{\frac{1}{2}m}k^m & m\in\{2,4,\ldots\}. \\ I_k(\kappa)\cos(k(\phi-\bar{\phi})), & \end{cases} \end{aligned} \tag{27}$$

One can see that when m is an odd number, the difference of both terms results in

$$\begin{aligned} &\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &= \begin{cases} \frac{1}{\pi I_0(\kappa)}2\sum_{k=1}^{\infty}(-1)^{\frac{1}{2}(m+1)}k^m I_k(\kappa) & m\in\{1,3,\ldots\}; \\ \cos(k\bar{\phi})\sin(k\pi), & \\ -\frac{1}{\pi I_0(\kappa)}2\sum_{k=1}^{\infty}(-1)^{\frac{1}{2}m}k^m I_k(\kappa) & m\in\{2,4,\ldots\}, \\ \sin(k\bar{\phi})\sin(k\pi), & \end{cases} \\ &= 0, \end{aligned} \tag{28}$$

which provides that the condition $\frac{\mathrm{d}^m}{\mathrm{d}\phi^m} p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} = 0$ holds for any nominal direction $\bar{\phi}$.

We find that only one notable angular distribution, the von Mises model [32], satisfies the condition. The cosine angular distribution model [7,27] is also tested for the condition and found to satisfy it; however, the cosine distribution model has never been regarded as a sound angular distribution model. In nutshell, the condition is beneficial only for one notable angular model, i.e., the von Mises distribution, where it leads to obtaining a simple formula for the SFC. For other distributions which are usually chosen as the popular spatial models [8,9,21,23,28]–[31], one will have to calculate the SFC through tedious computations without benefiting from the proposed condition.

V. Numerical Examples

The SFC computation for the Gaussian angular distribution needs to calculate the error functions with complex arguments in (11) and (13). Since an exact method to calculate such error functions with the complex arguments does not exist in the literature, a well-known approximation is given by (see [25, eq. (7.1.29)] and [39, eq. (5)])

$$\begin{aligned}\mathrm{erf}(x+\mathrm{j}y) &= \mathrm{erf}(x) + \frac{1}{2\pi x}\mathrm{e}^{-x^2}\left(1-\cos(2xy)+\mathrm{j}\sin(2xy)\right)\\ &+ \frac{1}{\pi}2\mathrm{e}^{-x^2}\sum_{\tilde{n}=1}^{\infty}\frac{1}{\tilde{n}^2+4x^2}\mathrm{e}^{-\frac{1}{4}\tilde{n}^2}\\ &(f_{\tilde{n}}(x,y)+\mathrm{j}g_{\tilde{n}}(x,y)) + \epsilon(x,y),\end{aligned} \tag{29}$$

where $f_{\tilde{n}}(x,y)$ and $g_{\tilde{n}}(x,y)$ are given by

$$\begin{aligned}f_{\tilde{n}}(x,y) &= 2x - 2x\cosh(\tilde{n}y)\cos(2xy)\\ &+ \tilde{n}\sinh(\tilde{n}y)\sin(2xy),\end{aligned} \tag{30a}$$

$$\begin{aligned}g_{\tilde{n}}(x,y) &= 2x\cosh(\tilde{n}y)\sin(2xy)\\ &+ \tilde{n}\sinh(\tilde{n}y)\cos(2xy),\end{aligned} \tag{30b}$$

and the approximation error $\epsilon(x,y)$ is on the order of $\frac{|\epsilon(x,y)|}{|\mathrm{erf}(x+\mathrm{j}y)|} \approx 10^{-16}$. Alternatively, an available MATLAB® package is developed in [40]. Due to the lack of the utility and the random number generation of the cosine distribution (see [19] and [20]), we omit the simulation for the cosine angular distribution. The Laplacian random number is calculated from [41, p. 94]. The von Mises random number is generated by a method presented in [42]. Since the correlation function in (19) is independent of the angular standard deviation σ_ϕ, any value of κ for the von Mises angular distribution is valid. However, we found that only a small value of κ substituted into [42] can provide accurate results to (19). For simplicity, we adopt $\kappa = 1$. In principle, the cosine and the von Mises angular distributions should correspond to (19).

We approximate the infinite summation in (3) by using only a finite number of the first 10^6 terms for the uniform and the Laplacian angular distributions and the first 20 terms for the Gaussian angular distribution. For the infinite summation in (29), the first 10^3 terms are added altogether. We found that the approach in (29) and the method in [40] provide nearly the same result in computing the complex error function.

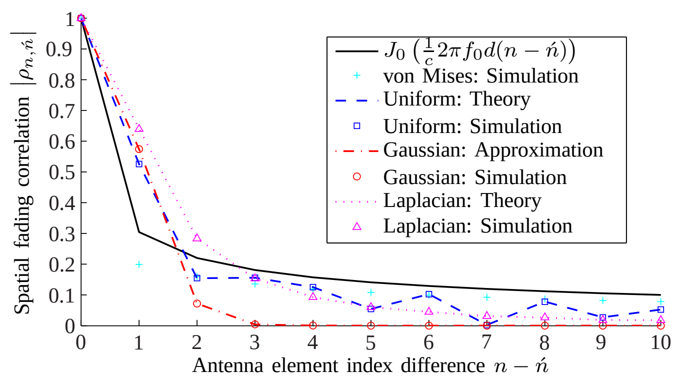

Fig. 1. The spatial fading correlation of several angular distributions as a function of antenna element index difference $n-\acute{n}$ for the nominal angle $\bar{\phi} = 0°$ and the angular standard deviation $\sigma_\phi = 20°$ with $N_\mathrm{R} = 10^7$ independent simulation runs.

However, the method in (29) fails in computing the SFC. The wave propagation speed is assigned as $c = 3\times10^8$ m/s. The operating frequency is assumed to be on the ultrawideband region, i.e., $f_0 = \frac{1}{2}(10.6+3.1)\times10^9$ Hz. The antenna element separation distance is chosen as a half of wave length, i.e. $d = \frac{1}{2}\left(\frac{c}{f_0}\right) = 0.0218978$ m. According to (2), the random evaluation of the SFC is chosen as the sample mean $\hat{\rho}_{n,\acute{n}} = \frac{1}{N_\mathrm{R}}\sum_{n_\mathrm{R}=1}^{N_\mathrm{R}} \mathrm{e}^{\frac{1}{c}\mathrm{j}2\pi f_0 d(n-\acute{n})\sin(\phi_{n_\mathrm{R}})}$, where N_R is the number of independent simulation runs.

In Fig. 1, we consider the SFC as a function of the difference of the antenna element indices. Note that the difference of the antenna indices $n-\acute{n}$ is valid for only the uniform linear array. For $n-\acute{n}=0$, the SFCs of all the angular distributions become unity, since the received signal at the same antenna element is fully correlated to itself. When the difference of the antenna element indices increases, the signals at different antenna elements are partially correlated. Hence, the SFC is lower than one. For the von Mises distribution, the simulation results coincide with the Jakes' model, since the von Mises and the cosine distributions follow the condition in (18), i.e., their sinusoid coefficients c_k and s_k are zeros. Theoretically, both angular models should provide the same SFC as the Jakes' model. Regarding the uniform distribution, the simulation results well coincide with the theoretical quantity. For the Laplacian angular distribution, the simulation results correspond to the theoretical derivation. Taking into account the Gaussian angular distribution, the numerical results of the SFC closely coincide with the theoretical results. However, the computation of the SFC involves the summations of two sinusoidal coefficients. In the case of the Gaussian angular distribution, each sinusoidal coefficient requires the summation of two terms, each of which is the error function with the complex argument. Since the error function is approximated with a finite number of summation terms, the remained infinite term in the complex-valued error function approximation leads to the cumulative errors in the final SFC computation. This approximation error results in the mismatches between the numerical results and the theoretical results, which are noticeable in the logarithm scale of the SFC for $n-\acute{n}>4$.

VI. Conclusion

A new perspective on the SFC is presented. The condition for obtaining a simple formula $J_0\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)$ for the SFC is exposed by the test of the angular distribution, i.e., $\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}p_\phi(\phi)\Big|_{\phi=-\pi}^{\phi=\pi}=0$. We have applied the proposition to the cosine, uniform, Gaussian, Laplacian, and von Mises distributions. It has been found that the cosine and the von Mises distributions absolutely lie in this kind of the distributions, whose SFC yields $J_0\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)$. The results of the proposition coincide with those of the classical method for the computation of the SFC.

Appendix A
Proof of Proposition 1

The proof is based on the by-part integrations for both c_k and s_k. In what follows, we show that $c_k=0$ and $s_k=0$.

Let $u_1=p_\phi(\phi)$ and $\mathrm{d}v_1=\cos(2k\phi)\mathrm{d}\phi$ be the variables of the integration by parts. Since $\mathrm{d}u_1=\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\mathrm{d}\phi$ and $v_1=\frac{1}{2k}\sin(2k\phi)$, we have

$$\begin{aligned} c_k &= \frac{1}{2k}\Bigg(p_\phi(\phi)\sin(2k\phi)\big|_{\phi=-\pi}^{\pi} \\ &\quad -\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\sin(2k\phi)\mathrm{d}\phi\Bigg) \\ &= -\frac{1}{2k}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\sin(2k\phi)\mathrm{d}\phi. \end{aligned} \tag{31}$$

Let $u_2=\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)$ and $\mathrm{d}v_2=\sin(2k\phi)\mathrm{d}\phi$ be the variables of the integration by parts. Since $\mathrm{d}u_2=\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\right)\mathrm{d}\phi$ and $v_2=-\frac{1}{2k}\cos(2k\phi)$, we have

$$\begin{aligned} c_k &= \left(-\frac{1}{2k}\right)^2\Bigg(\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\cos(2k\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\right)\cos(2k\phi)\mathrm{d}\phi\Bigg) \\ &= \left(-\frac{1}{2k}\right)^2\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\left(-\frac{1}{2k}\right)^2\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\right)\cos(2k\phi)\mathrm{d}\phi. \end{aligned} \tag{32}$$

Inferring from (31), we have

$$\begin{aligned} c_k &= \left(-\frac{1}{2k}\right)^2\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\left(-\frac{1}{2k}\right)^3\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^3}{\mathrm{d}\phi^3}p_\phi(\phi)\right)\sin(2k\phi)\mathrm{d}\phi. \end{aligned} \tag{33}$$

Inferring from (32), we have

$$\begin{aligned} c_k &= \left(-\frac{1}{2k}\right)^2\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi}-\left(-\frac{1}{2k}\right)^4\frac{\mathrm{d}^3}{\mathrm{d}\phi^3}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\left(-\frac{1}{2k}\right)^4\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^4}{\mathrm{d}\phi^4}p_\phi(\phi)\right)\cos(2k\phi)\mathrm{d}\phi. \end{aligned} \tag{34}$$

At the $2M$-th by-part integration for $M\in\{1,2,\ldots\}$, we have by induction

$$\begin{aligned} c_k &= \sum_{m=1}^{M}(-1)^{m-1}\left(-\frac{1}{2k}\right)^{2m}\frac{\mathrm{d}^{2m-1}}{\mathrm{d}\phi^{2m-1}}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\left(-\frac{1}{2k}\right)^{2M}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^{2M}}{\mathrm{d}\phi^{2M}}p_\phi(\phi)\right)\cos(2k\phi)\mathrm{d}\phi. \end{aligned} \tag{35}$$

If $p_\phi(\phi)$ is differentiable up to $2M$ times, the term $\frac{\mathrm{d}^{2M}}{\mathrm{d}\phi^{2M}}p_\phi(\phi)$ remains constant. We have

$$\begin{aligned} c_k &= \sum_{m=1}^{M}(-1)^{m-1}\left(-\frac{1}{2k}\right)^{2m}\frac{\mathrm{d}^{2m-1}}{\mathrm{d}\phi^{2m-1}}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\left(-\frac{1}{2k}\right)^{2M}\left(\frac{\mathrm{d}^{2M}}{\mathrm{d}\phi^{2M}}p_\phi(\phi)\right)\int_{-\pi}^{\pi}\cos(2k\phi)\mathrm{d}\phi \\ &= \sum_{m=1}^{M}(-1)^{m-1}\left(-\frac{1}{2k}\right)^{2m}\frac{\mathrm{d}^{2m-1}}{\mathrm{d}\phi^{2m-1}}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi}. \end{aligned} \tag{36}$$

It can be seen that if $\frac{\mathrm{d}^{2m-1}}{\mathrm{d}\phi^{2m-1}}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi}$ is zero for all $m\in\{1,2,\ldots,M\}$, we have $c_k=0$.

Let $u_1=p_\phi(\phi)$ and $\mathrm{d}v_1=\sin((2k-1)\phi)\mathrm{d}\phi$ be the variables of the integration by parts. Since $\mathrm{d}u_1=\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\mathrm{d}\phi$ and $v_1=-\frac{1}{2k-1}\cos((2k-1)\phi)$, we have

$$\begin{aligned} s_k &= -\frac{1}{2k-1}\Bigg(p_\phi(\phi)\cos((2k-1)\phi)\big|_{\phi=-\pi}^{\pi} \\ &\quad -\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\cos((2k-1)\phi)\mathrm{d}\phi\Bigg) \\ &= \frac{1}{2k-1}\left(p_\phi(\pi)-p_\phi(-\pi)\right) \\ &\quad +\frac{1}{2k-1}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\cos((2k-1)\phi)\mathrm{d}\phi. \end{aligned} \tag{37}$$

Let $u_2=\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)$ and $\mathrm{d}v_2=\cos((2k-1)\phi)\mathrm{d}\phi$ be the variables of the integration by parts. Since $\mathrm{d}u_2=\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\right)\mathrm{d}\phi$ and $v_2=\frac{1}{2k-1}\sin((2k-1)\phi)$, we have

$$\begin{aligned} s_k &= \frac{1}{2k-1}\left(p_\phi(\pi)-p_\phi(-\pi)\right) \\ &\quad +\frac{1}{(2k-1)^2}\Bigg(\left(\frac{\mathrm{d}}{\mathrm{d}\phi}p_\phi(\phi)\right)\sin((2k-1)\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\right)\sin((2k-1)\phi)\mathrm{d}\phi\Bigg) \\ &= \frac{1}{2k-1}\left(p_\phi(\pi)-p_\phi(-\pi)\right) \\ &\quad -\frac{1}{(2k-1)^2}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\right)\sin((2k-1)\phi)\mathrm{d}\phi. \end{aligned} \tag{38}$$

Inferring from (37), we have

$$\begin{aligned} s_k &= \frac{1}{2k-1}\left(p_\phi(\pi)-p_\phi(-\pi)\right)-\frac{1}{(2k-1)^3}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi} \\ &\quad -\frac{1}{(2k-1)^3}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^3}{\mathrm{d}\phi^3}p_\phi(\phi)\right)\cos((2k-1)\phi)\mathrm{d}\phi. \end{aligned} \tag{39}$$

Inferring from (38), we have

$$s_k = \frac{1}{2k-1}\left(p_\phi(\pi) - p_\phi(-\pi)\right) - \frac{1}{(2k-1)^3}\ \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}p_\phi(\phi)\bigg|_{\phi=-\pi}^{\pi} + \frac{1}{(2k-1)^4}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^4}{\mathrm{d}\phi^4}p_\phi(\phi)\right)\sin((2k-1)\phi)\mathrm{d}\phi. \tag{40}$$

At the $2M$-th by-part integration for $M \in \{1, 2, \ldots\}$, we have by induction

$$s_k = \sum_{m=1}^{M}(-1)^{m-1}\left(\frac{1}{2k-1}\right)^{2m-1}\ \frac{\mathrm{d}^{2(m-1)}}{\mathrm{d}\phi^{2(m-1)}}p_\phi(\phi)\bigg|_{\phi=-\pi}^{\pi} + \frac{1}{(2k-1)^{2M}}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}^{2M}}{\mathrm{d}\phi^{2M}}p_\phi(\phi)\right)\sin((2k-1)\phi)\mathrm{d}\phi. \tag{41}$$

If $p_\phi(\phi)$ is differentiable up to $2M$ times, the term $\frac{\mathrm{d}^{2M}}{\mathrm{d}\phi^{2M}}p_\phi(\phi)$ remains constant. We have

$$\begin{aligned} s_k &= \sum_{m=1}^{M}(-1)^{m-1}\left(\frac{1}{2k-1}\right)^{2m-1}\ \frac{\mathrm{d}^{2(m-1)}}{\mathrm{d}\phi^{2(m-1)}}p_\phi(\phi)\bigg|_{\phi=-\pi}^{\pi} \\ &\quad + \frac{1}{(2k-1)^{2M}}\left(\frac{\mathrm{d}^{2M}}{\mathrm{d}\phi^{2M}}p_\phi(\phi)\right)\int_{-\pi}^{\pi}\sin((2k-1)\phi)\mathrm{d}\phi \\ &= \sum_{m=1}^{M}(-1)^{m-1}\left(\frac{1}{2k-1}\right)^{2m-1}\ \frac{\mathrm{d}^{2(m-1)}}{\mathrm{d}\phi^{2(m-1)}}p_\phi(\phi)\bigg|_{\phi=-\pi}^{\pi}. \end{aligned} \tag{42}$$

It can be seen that if $\frac{\mathrm{d}^{2(m-1)}}{\mathrm{d}\phi^{2(m-1)}}p_\phi(\phi)\Big|_{\phi=-\pi}^{\pi}$ is zero for all $m \in \{1, 2, \ldots, M\}$, we have $s_k = 0$.

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